Semianalytical transient solution of a delayed differential equation and its application to the tracking motion in the sensory-motor system

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(Received 10 June 2006; revised manuscript received 5 October 2006; published 8 January 2007)

We derived semianalytically the transient solution of a delayed differential equation that had been shown to be a simple but good model of the sensory-motor system. In the present Brief Report, we applied this transient solution for studying the global nature of the transient tracking motion when visual target information is changed suddenly. The results clarified that the dynamic error minimization principle in hand motion observed experimentally is robust over a wide range of the parameter space of the delay time, the time constant, and the feedforward parameter.

DOI: 10.1103/PhysRevE.75.012901 PACS number(s): 87.19.St, 02.30.Ks, 87.19.Bb

The sensory-motor system of humans as well as other animals is controlled by neural systems which have an inevitable delay, without some compensation mechanism in the brain which generates appropriate behavior despite the delay [1–5]. Recently, the authors demonstrated by psychophysical experiments [6,7] involving a transient hand-tracking task that the human visual-motor system operates not only predictively but also "proactively" for harmonic stimuli of a finite frequency region. Proactive control means a brain function by which motion precedes stimulus by an amount needed to minimize the error adjusting to changes in speed of the stimulus [6,7] (Fig. 1).

There have been a number of studies on the behavior of the sensory-motor system using delayed differential equations, including visual tracking [8,9], auditory synchronization [10], postural control [11], and pole balancing [12], etc. The present authors also demonstrated by numerical simulation that a model differential equation which includes a feedforward term with delay in addition to the feedback term with delay showed proactive behavior, i.e., the dynamic error minimization principle.

Robustness of the proactive behavior was also shown in the hand-tracking task experiments in the previous paper; both the amount of precedence of the hand in steady state and the optimum value for the transient error varied from subject to subject, interestingly with a strong correlation. Therefore, it should be valuable to verify this robustness by an analytical method, in spite of the preliminary evidence by numerical simulation. Furthermore, analytical studies will contribute to the theoretical understanding of the sensory-motor system to know if and why a type of delayed differential equation with feedforward control mechanism robustly shows proactive behavior. In this paper we derive by a perturbation method a transient solution of the delayed differential equation that we used for the simulation to compare with experimental results in the previous paper, and show in fact that the solution demonstrates a proactive nature in a wide parameter space.

The simplest expression of a delayed equation [13] including a feedback and a feedforward term is

$$\dot{X}(t) = \frac{1}{\tau}[T(t-\delta) - X(t-\delta)] + \gamma\dot{T}(t-\delta). \tag{1}$$

Here, T and X denote the coordinates of the external stimulus and the controlled signal, respectively, where τ is the time constant of the system and δ is the delay time of signal

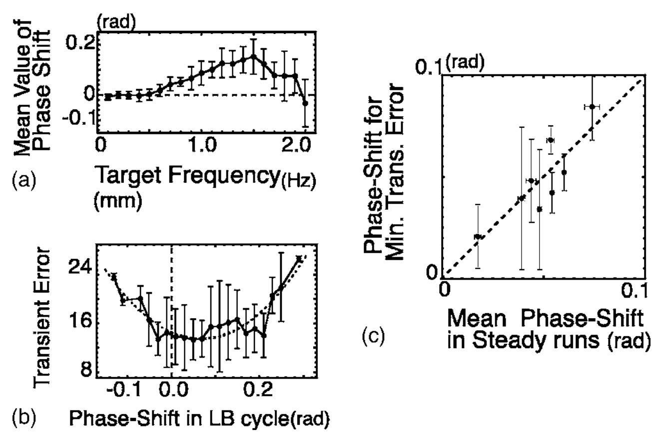

FIG. 1. Overview of hand-tracking experiments (from Ref. [6]). (a) Frequency dependence of the mean phase lead of the hand with respect to the target which moves sinusoidally with a given frequency. For example, the phase lead is 0.06 within the error of 0.02 rad at 0.8 Hz. (b) An example of the transient error of the hand position with respect to the target position for a subject with initial frequency 0.8 Hz averaged over the final frequency of transition, measured as a function of the phase-lead quantity at the last cycle before the transition (LB cycle). The graph shows a minimum of the transient error at around the phase lead of 0.06 rad. (c) Correlation between the mean phase lead in a steady run and the optimum phase lead in the LB cycle for minimum transition error for eight subjects.

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transport. This equation comprises a delayed error feedback term and a speed feedforward term of the stimulus. In addition, γ represents the relative contribution of the feedforward information to the feedback information. In hand-tracking tasks, T and X represent the target motion and the hand motion, respectively.

The steady phase shift ψ of Eq. (1) for a sinusoidal target motion $T(t)=\exp(i\omega t)$ is expressed as follows:

$$\psi = \tan^{-1}\left(\frac{\gamma\omega\tau - \omega\tau\cos\omega\delta - \gamma\omega^2\tau^2\sin\omega\delta}{1-\omega\tau\sin\omega\delta+\gamma\omega^2\tau^2\cos\omega\delta}\right). \quad (2)$$

In a psychophysical experiment [Fig. 1(a)], the mean value of the steady phase shift is positive, and it increases with the target frequency up to a certain frequency and decreases abruptly thereafter. By expanding Eq. (2) with respect to the target frequency, we found out that the first- and third-order terms in the numerator and the second-order term in the denominator of $\omega\tau$ and/or $\omega\delta$ should be positive, negative, and positive, respectively, in order to reproduce these phase-lead behaviors in the steady state. These conditions for γ are expressed as follows:

$$\gamma - 1 > 0 \quad \text{and} \quad \gamma - \delta/\tau > 0. \quad (3)$$

The transient experiment was designed such that the target frequency is kept at f_1 until $t=0$ when it is switched to f_2. We write the transient solution in the form of Eq. (4),

$$X(t) = \xi_1(t)X_1(t) + [1-\xi_2(t)]X_2(t) \quad (t \geq \delta), \quad (4)$$

$$\xi_i(\delta) = 1, \quad \xi_i(+\infty) = 0 \quad (i=1,2), \quad (5)$$

where $X_1(t)=\exp[i(\omega_1 t+\psi_1)]$ and $X_2(t)=\exp[i(\omega_2 t+\psi_2)]$ are the stationary solutions of the hand motion with the target frequencies f_1 and f_2, respectively. The expression in this form is similar to the formalism of the time-dependent perturbation theory used in quantum mechanics. Note that the transient hand motion is $X_1(t)$ from $t=0$ to $t=\delta$ because of causality.

By inserting Eq. (4) into Eq. (1), we obtained the following equation:

$$\mathbf{A}[\Xi_1(t) + \Xi_2(t)] = 0, \quad (6)$$

where $\mathbf{A}=\tau d/dt+\mathbf{D}$ is a linear operator with $\mathbf{D}$ as a delay operator where $\mathbf{D}x(t)=x(t-\delta)$, $\Xi_1(t)=\xi_1(t)X_1(t)$, and $\Xi_2(t)=-\xi_2(t)X_2(t)$. Equation (6) means that $\Xi_1(t)+\Xi_2(t)$ is the eigenfunction of $\mathbf{A}$ with eigenvalue 0. However, the eigenfunction of the operator $\mathbf{A}$ is of the type $\xi(t) =\exp[(-p+iq)(t-\delta)]$, so Ξ_1 and Ξ_2 must independently be the eigenfunction of the linear operator $\mathbf{A}$, apart from the coefficients which are determined by the continuity conditions.

The values of p and Q $(=q+\omega_i)$ are obtained from the following equations:

$$(\tau p)^2 + (\tau Q)^2 = e^{2p\delta}, \quad (7)$$

$$Q/p = \tan Q\delta. \quad (8)$$

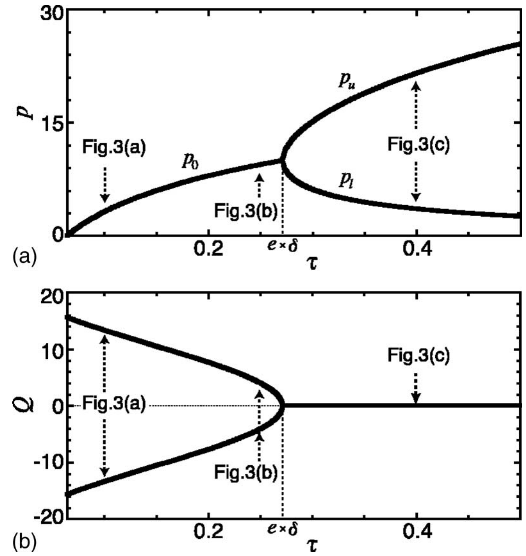

FIG. 2. Solutions of Eqs. (7) and (8) for p and Q at $\delta=0.1$. (a) p, where p_0 is the solution for $\tau/\delta \leq e$, and p_u and p_l are the solutions for $\tau/\delta > e$. (b) Q. Here, the notations Figs. 3(a)–3(c) indicate the conditions for Fig. 3.

No time scale argument appears in this deduction. Figures 2(a) and 2(b) show the values of p and Q that satisfy Eqs. (7) and (8) at $\delta=0.1$. When $\tau/\delta \leq e$, p $(=p_0)$ increases with τ up to $=1/\delta$. On the other hand, when $\tau/\delta > e$, p has two values. One $(=p_u)$ is greater than $1/\delta$, and the other $(=p_l)$ is less than $1/\delta$. Here, Q is always zero. Equations (7) and (8) allow a negative value for p. However, negative p does not satisfy the boundary condition given by Eq. (5).

Thus, $\xi_i(t)$ is expressed as follows:

$$\xi_i(t) = \begin{cases} a_i e^{[-p_0 - i(\omega_i - Q)](t-\delta)} + b_i e^{[-p_0 - i(\omega_i + Q)](t-\delta)} & (\tau/\delta \leq e), \\ c_i e^{(-p_u - i\omega_i)(t-\delta)} + d_i e^{(-p_l - i\omega_i)(t-\delta)} & (\tau/\delta > e). \end{cases} \quad (9)$$

The values of a_i, b_i, c_i, and d_i are determined by the position and velocity continuity conditions at $t=\delta$.

Figure 3 demonstrates some examples of the semianalytical transient solution together with the numerical solution for choices of initial and final target frequencies. Figures 3(a) and 3(b) illustrate examples for the condition $\tau/\delta \leq e$, and Fig. 3(c) shows an example for the condition $\tau/\delta > e$.

All of the results show that the solution obtained by the present perturbation method is similar to that obtained by direct numerical simulation under any conditions. That is, the semianalytical transient solution obtained by using the present perturbation analysis is a good approximation of the transient behavior in hand tracking.

In the following we calculate the transient error of the hand motion with respect to the target motion just after the frequency jump by using the semianalytical transient solution [Eq. (4)]. The time duration from the frequency change to rest in the stationary state can be regarded as the time constant ($1/p_0$ or $1/p_l$) of the coefficient $\xi(t)$. The expected

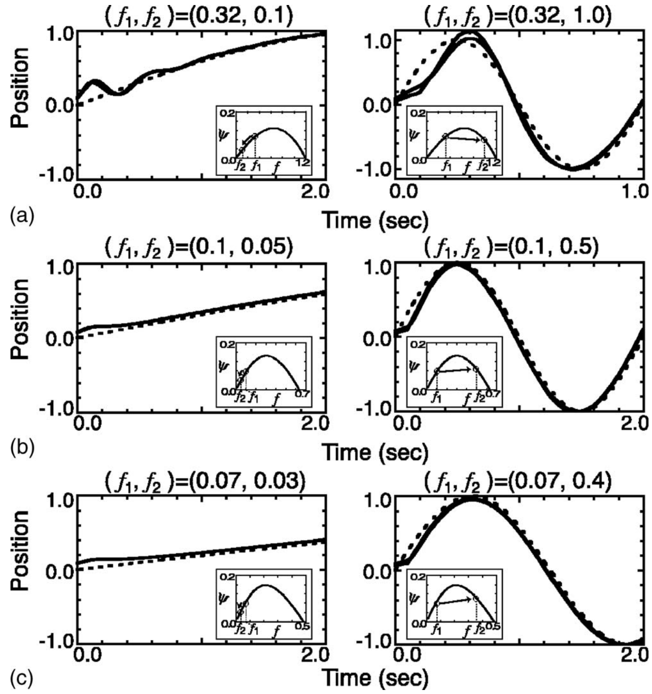

FIG. 3. Results of a comparison between the analytical and numerical solutions. The thick black solid line and the thin gray solid line represent the analytical solution [Eq. (4)] and the numerical solution of Eq. (1), respectively. The dotted line represents the target motion. τ=(a) 0.1, (b) 0.25, and (c) 0.4. The values of γ and δ are 1.5 and 0.1, respectively. The insets illustrate the frequency condition for each case. The curve in the inset represents the target frequency (f) dependence of the steady phase shift (ψ).

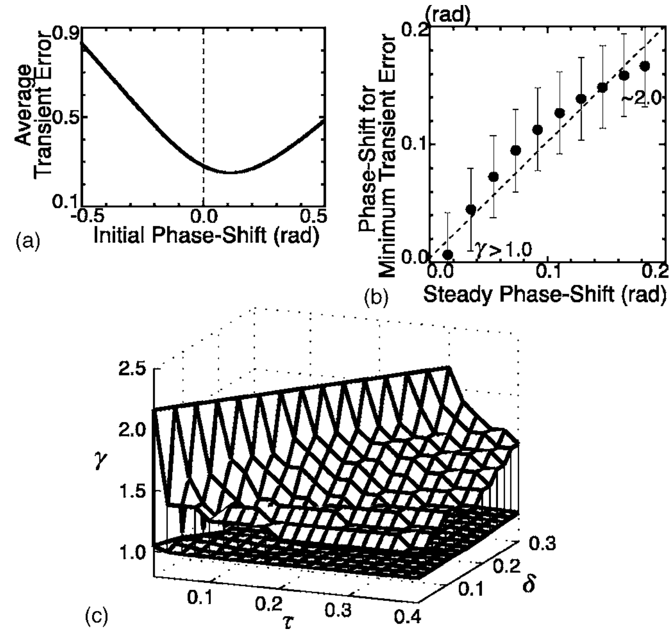

FIG. 4. (a) Transient error as a function of the initial phase. The initial frequency (f_1) is 0.32 Hz. The values of γ, τ, and δ are 1.5, 0.1, and 0.1, respectively. We varied the initial phase shift in simulation and calculated the transient error by Eq. (10). Then we selected the initial phase shift which minimizes the transient error (the optimum phase shift). (b) Correlation between the optimum initial phase shift thus obtained and the steady phase shift obtained by Eq. (2) for difference values of γ ($1.1 \leq \gamma \leq 2.0$). The initial frequency is the same as that for (a). The error bar is 0.035 rad, which is the maximum estimated error in the hand-tracking experiment [Fig. 1(c)]. The values of τ and δ are 0.1 and 0.1, respectively. (c) Parameter space in which proactive control is guaranteed. Here, $f_1 = \bar{f}_{pv}/2$ [$\bar{f}_{pv} = (1/4)\int_1^5 f_{pv}(\gamma) d\gamma$], where f_{pv} denotes the frequency at which the phase shift with precedence takes peak values.

transient error was calculated as the average value over the final frequency. The following transient error ($\mathcal{E}_T$) is used for comparison with the hand-tracking experiments:

$$\mathcal{E}_T = \int_0^{f_c} \left(p \int_0^{1/p} |T(t) - X(t)|^2 dt \right)^{1/2} \mathrm{Pr}(f_2) df_2, \qquad (10)$$

where $\mathrm{Pr}(f_2)$ is the probability that the final state frequency is f_2, and f_c denotes the cutoff frequency in the phase shift profile.

Figure 4(a) shows the transient error as a function of the initial phase ψ_{ini} where we assume that $\mathrm{Pr}(f_2)$ is a constant. The error curve calculated by Eq. (10) exhibits a parabolic characteristic with the minimum at a small positive phase. This minimum can be determined easily with the estimated error. Furthermore, the profile of the error curve is in good agreement with that of the experimental results [Fig. 1(b)].

Figure 4(b) illustrates the correlation between the steady phase shift before the transition and the phase shift corresponding to the minimum transient error for the various values of γ. The present results show that these two quantities correlate for a wide range of γ of the delayed equation within the numerical error. The steady phase shift corresponds to the optimum phase shift to achieve the minimum transient error within the maximum estimated error in the hand-tracking experiments [Fig. 1(c)]. The delayed feedforward (DFF) model [Eq. (1)] reproduces the dynamic error minimization principle of proactive control. Furthermore, this correlation profile is similar to that of the experimental result. The phase-lead value for each subject differs from that of other subjects. The results shown in Fig. 4(b) suggest that this discrepancy among the subjects can be represented by the difference in the value of γ, which is the strength of the contribution of the target speed recognition in a visual system.

Figure 4(c) shows the parameter space of (γ, τ, δ) such that the optimum phase shift corresponds to the steady phase shift within the maximum estimated error (0.035 rad). We can see that there is a wide proactive region when τ is greater than δ. In this region $\gamma > 1$ is the only condition for positive steady phase shift [Eq. (3)]. When τ is in the vicinity of δ, proactive control is guaranteed in the range of γ between 1.0 and approximately 2.1. On the other hand, when τ is far from δ, the range of γ is reduced between 1.0 and 1.1. This result indicates that systems in which τ approaches δ have the ability to robustly employ proactive control to minimize the transient error.

In the present Brief Report, we have clarified the following two points. (1) A semianalytical transient solution of a delayed differential equation derived by a perturbation method was found to show an excellent agreement with direct numerical simulation. (2) Using this transient solution it was possible to demonstrate the robustness of proactive behavior of the model equation for a wide range of parameter space, suggesting that the delayed differential equation with feedforward term is useful for studying the sensory-motor system.

The present study was partly supported by the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Young Scientists (B) No. 18700196, and the Cooperative Research Project of the Research Institute of Electrical Communication, Tohoku University.

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